RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. SIXTH SEMESTER EXAMINATION, MAY 2017

THIRD YEAR [BATCH 2014-17]

Date : 04/05/2017 **MATHEMATICS (Honours)**

Time : 11 am – 3 pm **Paper :** VII Full Marks : 100

[Use a separate Answer Book for each group]

Group – A

Answer any three questions from Question Nos. 1 to 5 : [3×10]

1. a) Find the Fourier series for the function f on $(-\pi,\pi]$ where

$$f(x)=\begin{cases}-1 & \text{if} \quad -\pi<x<0\\ 1 & \text{if} \quad 0\le x\le \pi\end{cases}$$

Hence, deduce the value of $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}$. [4+2]

b) Verify Gauss' theorem for the surface integral $\iint_S (x^2-yz)dydz+(y^2-zx)dzdx+(z^2-xy)dxdy$, where S is the surface of the rectangular parallelopiped $0\le x\le a$, $0\le y\le b$, $0\le z\le c$. [4]

2. a) Discuss the convergence of $\Gamma(n)=\int_0^{\infty} e^{-x}x^{n-1}dx$. [4]

b) Using Gauss' theorem, show that the value of $\iint_S xdydz+ydzdx+zdxdy$ is 81π, where S is the surface of the region bounded by the cylinder $x^2+y^2=9$ and the planes z = 0 and z = 3. [4]

c) Show that $B(x,y)=2\int_0^{\pi/2}\sin^{2x-1}\theta\cos^{2y-1}\theta\, d\theta$ for x,y > 0. [2]

3. a) Obtain the Fourier series expansion of f(x) = x sinx on $[-\pi,\pi]$. Hence, deduce that

$$\frac{\pi}{4}=\frac{1}{2}+\frac{1}{1\cdot 3}-\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}-\ldots.$$ [4+2]

b) Compute the volume V, common to the ellipsoid of revolution $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and the cylinder $x^2+y^2-ay=0$. [4]

4. a) Prove that $\sqrt{\pi}\,\Gamma(2x)=2^{2x-1}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)$, for x > 0. [4]

b) Prove that $\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta+c\sin\theta}=\frac{2\pi}{\sqrt{a^2-r^2}}$, where $r^2=b^2+c^2$ and r < a. [3]

c) Show that $\int_0^1 dx\int_0^{\sqrt{(1-x^2)}}\frac{dy}{(1+e^y)\sqrt{(1-x^2-y^2)}}=\frac{\pi}{2}\log\left[\frac{2e}{1+e}\right]$. [3]

5. a) Show that $\int_0^{\pi/2}\log\sin x\,dx$ is convergent and evaluate it. [5]

b) Prove that $\int_0^\infty \frac{x \log x}{(1+x^2)^2} dx$ is convergent and converges to 0. [5]

Answer <u>any two</u> questions from <u>Question Nos. 6 to 8</u> : [2×10]

6. a) Prove that p is prime iff $(p-1)!+1 \equiv 0 \pmod p$. [5]

   b) If $f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right)$ for every positive integer n, then prove that $F(n) = \sum_{d|n} f(d)$. [5]

7. a) If n > 2, prove that $\phi(n)$ is even. [3]

   b) Define μ function. Prove that the function μ is multiplicative. [3]

   c) Find the smallest positive integer x such that $x \equiv 3 \pmod 5$, $x \equiv 4 \pmod 7$ and $x \equiv 2 \pmod 3$. [4]

8. a) If p is prime and p|ab then prove that either p|a or p|b. [3]

   b) If p and p^2+8 are both prime numbers, prove that p = 3. [3]

   c) Use the fundamental theorem of arithmetic to prove that if p is prime, then $\sqrt{p}$ is not a rational number. [4]

Group – B

Answer <u>any three</u> questions from <u>Question nos. 9 to 13</u> : [3×10]

9. a) State the axiomatic definition of probability. Show that conditional probabilities satisfy all the three axioms of probability. [2+4]

   b) For any n events of a random experiment E prove that $P(A_1 A_2 ... A_n) \geq 1 - \sum_{i=1}^{n} P(\bar{A}_i)$. [4]

10. a) If a die is thrown K-times, show that the probability of an even number of sixes is $\frac{\left[1+\left(\frac{2}{3}\right)^K\right]}{2}$. [5]

    b) The probability that a product produced by a machine to be defective is 0·01. If 50 products are taken at random, find the probability that exactly 3 of them will be defective. Approximate it by Poisson distribution and evaluate the error of approximation. [5]

11. a) The distribution function F(x) of a random variable x is defined as follows :

    $$\begin{aligned} F(x) &= A \quad ; \quad -\infty < x < -1 \\ &= B \quad ; \quad -1 \leq x < 0 \\ &= C \quad ; \quad 0 \leq x < 2 \\ &= D \quad ; \quad 2 \leq x < \infty \end{aligned}$$

    where A, B, C, D are constants. Determine the values of A,B,C,D; given that $P(x=0) = \frac{1}{6}$, $P(x>1) = \frac{2}{3}$. [5]

    b) If X is a binomial (n,p) variate, then prove that $\mu_{K+1} = p(1-p)\left(nK\mu_{K-1} + \frac{d\mu_K}{dp}\right)$, where μ_K is the K-th central moment. [5]

12. a) If X is a Poisson variate with parameter μ, show that $P(x \le n) = \frac{1}{\lfloor n} \int_{\mu}^{\infty} e^{-x} x^n dx, n \in \mathbb{N}$. [4]

b) If X, Y be two random variables such that $E(X^2)$, $E(Y^2)$, E(XY) exist, then show that $\{E(XY)\}^2 \le E(X^2)E(Y^2)$. Hence deduce that $-1 \le \rho(X, Y) \le 1$. [4+2]

13. a) State the Tchebycheff's inequality.

A random variable X has probability density function $12x^2(1-x)$, $0 < x < 1$.

Compute $P(|x-m| \ge 2\sigma)$ and compare with the limit given by Tchebyeheff's inequality. [2+4]

b) If the distribution of a random variable X is standard normal, then show that the random variable X^2 is a $\chi^2(1)$ variate. [4]

Answer <u>any two</u> questions from <u>Question nos. 14 to 16</u> : [2×10]

14. a) State and prove the Cauchy-Riemann equation. [5]

b) Prove that the set of zeros of an analytic function is isolated. [5]

15. a) Show that an analytic function with constant modulus is a constant. [4]

b) Let G be a region and define $G^* = \{Z \mid \bar{Z} \in G\}$. If $f: G \to \mathbb{C}$ is analytic, prove that $f^*: G^* \to \mathbb{C}$ defined by $f^*(Z) = f(\bar{Z})$ is also analytic. [4]

c) Prove that the series $\sum_{n \ge 1} n^{-z}$ converges in $D = \{Z \in \mathbb{C} : \text{Re}\, Z > 1\}$. [2]

16. a) Find the domain of convergence of the power series $\left(\frac{iz-1}{2+i}\right)^n$. [2]

b) Find all harmonic functions of the type $u = \phi(\sqrt{x^2 + y^2})$ that are not continuous. [5]

c) Give an example of a function $f(Z) = u + iv$, for which u, v satisfy Cauchy-Riemann equation but f is not differentiable. [3]

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